Degeneration of Orlik-Solomon algebras and Milnor fibers of complex line arrangements

Pauline Bailet

Hokkaido University

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Reference

Joint work with M. Yoshinaga

Degeneration of Orlik-Solomon algebras and Milnor fibers of complex line arrangements, Geometriae Dedicata, 2014, 10.1007/s10711-014-0027-7

The objects

Let $\bar{\mathcal{A}} = \{\bar{L}_0, \bar{L}_1, \ldots, \bar{L}_n\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a projective line arrangement.

Let α_j be the linear form defining $\bar{L}_j$, i.e.

$$\bar{L}_j := \{(x : y : z) \in \mathbb{P}^2_{\mathbb{C}} \mid \alpha_j(x, y, z) = 0\},$$

and $Q(x, y, z) = \prod_{j=0}^{n} \alpha_j(x, y, z)$ be the defining polynomial of $\bar{\mathcal{A}}$.

The objects

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- **The intersection lattice** $L(\bar{\mathcal{A}}) = \{\cap_{j \in I} \bar{L}_j\}$ codifies the combinatorics.
- **The complement** $M(\bar{\mathcal{A}}) = \mathbb{P}^2_{\mathbb{C}} \backslash \cup_{j=0}^{n} \bar{L}_j$.
- **The Milnor fiber** $F = \{(x, y, z) \in \mathbb{C}^3 \mid Q(x, y, z) = 1\} \subset \mathbb{C}^3$.

The objects

Let h be the **monodromy** of the Milnor fibration

$$\begin{array}{rccc} h: & F & \longrightarrow & F \\ & (x,y,z) & \longmapsto & \lambda \cdot (x,y,z), \end{array}$$

where $\lambda = exp(2\sqrt{-1}\pi/n+1)$,

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$$h^* : H^*(F,\mathbb{C}) \longrightarrow H^*(F,\mathbb{C}).$$

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We have the following decomposition

$$H^*(F,\mathbb{C}) = \bigoplus_{\beta^{n+1}=1} H^*(F,\mathbb{C})_\beta,$$

where $H^*(F,\mathbb{C})_\beta = ker\{h^* - \beta Id\}$.

Open questions

- Are the $H^*(F, \mathbb{C})$ determined by the arrangement's combinatorics?
- Are the h^* determined by the arrangement's combinatorics?

Notation: Let $k > 1$ be an integer. We denote by $\mu(\bar{L}_j, k)$ the number of intersection points on $\bar{L}_j$ with multiplicities divisible by k.

Motivation

Theorem (Libgober, 2002)

Let $k > 1$ and $\beta \neq 1$ be a non trivial eigenvalue of order k.

If $\mu(\bar{L}_j, k) = 0$ for some $\bar{L}_j \in \bar{\mathcal{A}}$, then $H^1(F, \mathbb{C})_\beta = 0$.

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Theorem (Yoshinaga, 2013)

Assume that $\bar{\mathcal{A}}$ is defined over $\mathbb{R}$.
Let $k > 1$ and $\beta \neq 1$ be a non trivial eigenvalue of order k.

$$\text{If } \mu(\bar{L}_j, k) \leq 1 \text{ for some } \bar{L}_j \in \bar{\mathcal{A}}, \text{ then } H^1(F, \mathbb{C})_\beta = 0.$$

Main vanishing result

Theorem (Yoshinaga, B.)

Let $\beta \neq 1$ be a non trivial eigenvalue of order p^s, p ***prime****, $s \geq 1$. Assume that $\bar{\mathcal{A}}$ is essential.*

$$\textit{If } \mu(\bar{L}_j, p) \leq 1 \textit{ for some } \bar{L}_j \in \bar{\mathcal{A}}, \textit{ then } H^1(F, \mathbb{C})_\beta = 0.$$

Orlik Solomon Algebra

We consider $\mathcal{A} = \{L_1, \dots, L_n\} \subset \mathbb{C}^2$ the **deconing** of $\bar{\mathcal{A}}$.

Let R be a commutative ring.

Let $A^*_R(\mathcal{A}) \simeq H^*(M(\mathcal{A}), R)$ be the **Orlik-Solomon algebra** of $\mathcal{A}$.

Let $\omega_1 = e_1 + e_2 + \cdots + e_n \in A^1_R(\mathcal{A})$, where $e_j = \frac{1}{2\sqrt{-1}\pi} \cdot \frac{d\alpha_j}{\alpha_j}$.

We consider the **Aomoto complex**:

$$(A^*_R(\mathcal{A}), \omega_1 \wedge) = \{\ A^*_R(\mathcal{A}) \xrightarrow{\omega_1 \wedge} A^{*+1}_R(\mathcal{A})\ \}_{*\geq 0}.$$

Key results

Theorem (Papadima, Suciu, 2010)

Let $p \in \mathbb{Z}$ be a prime, and $\beta \neq 1$ be an eigenvalue of order p^s, $s \geq 1$. Then

$$\dim H^1(F, \mathbb{C})_\beta \leq \dim H^1(A^*_{\mathbb{F}_p}(\mathcal{A}), \omega_1 \wedge).$$

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Total degeneration

Let $\mathcal{A} = \{L_1, \ldots, L_n\} \subset \mathbb{C}^2$ be the deconing of $\mathcal{A}$ and its partition in t classes of parallel lines:

$$\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_t.$$

Theorem (Yoshinaga, B.)

There exists a surjective homomorphism, called ***total degeneration****:*

$$\Delta_{tot} : A_R^*(\mathcal{A}) \longrightarrow A_R^*(\mathcal{C}_t),$$

where $\mathcal{C}_t$ is a central arrangement of t lines in $\mathbb{C}^2$.

Directional degeneration

Let $\mathcal{A} = \{L_1, \ldots, L_n\} \subset \mathbb{C}^2$ be the deconing of $\mathcal{A}$ and its partition in t classes of parallel lines:

$$\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_t.$$

Let us fix a class $\mathcal{A}_\alpha$. Assume $|\mathcal{A}_\alpha| = r$.

Theorem (Yoshinaga, B.)

There exists surjective homomorphism, called ***directional degeneration*** *with respect to* $\mathcal{A}_\alpha$

$$\Delta_{dir} : A_R^*(\mathcal{A}) \longrightarrow A_R^*(\mathcal{P}_r),$$

where $\mathcal{P}_r$ *is composed of* r *parallel lines and an other line transversal to them.*

Thank you for your attention!