Degeneration of Orlik-Solomon algebras and Milnor fibers of complex line arrangements

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Reference

Joint work with Masahiko Yoshinaga (Hokkaido)

Degeneration of Orlik-Solomon algebras and Milnor fibers of complex line arrangements, arxiv:1312.1771

The objects

Let $\bar{\mathcal{A}} = \{\bar{H}_0, \bar{H}_1, \ldots, \bar{H}_d\}$ be an arrangement of $d+1$ lines in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$, with complement

$$M(\bar{\mathcal{A}}) = \mathbb{P}^2_{\mathbb{C}} \setminus \cup_{j=0}^{d} \bar{H}_j.$$

Let α_j be a defining linear form of $\bar{H}_j$ and $Q(x, y, z) = \prod_{j=0}^{d} \alpha_j$ be the defining equation of $\bar{\mathcal{A}}$ (homogeneous, degree $d+1$ polynomial).

Let F be the Milnor fiber of $\bar{\mathcal{A}}$

$$F = \{(x, y, z) \in \mathbb{C}^3 \mid Q(x, y, z) = 1\} \subset \mathbb{C}^3.$$

The objects

$\bar{\mathcal{A}} = \{\bar{H}_0, \bar{H}_1, \dots, \bar{H}_d\} \subset \mathbb{P}^2_{\mathbb{C}}$.

With the identification $\mathbb{P}^2_{\mathbb{C}} \setminus \bar{H}_0 = \mathbb{C}^2$, let $\mathcal{A}$ be the deconing of $\bar{\mathcal{A}}$

$$\mathcal{A} = \{H_1, \dots, H_d\} \subset \mathbb{C}^2,$$

where $H_j = \bar{H}_j \cap \mathbb{C}^2$.

We denote its complement by

$$M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{j=1}^{d} H_j = M(\bar{\mathcal{A}}).$$

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The objects

Let $\lambda = exp(2i\pi/(d+1))$.

Let h be the monodromy of the Milnor fibration

$$\begin{aligned} h : F &\to F \\ x &\mapsto \lambda x \end{aligned}$$

and h^q the induced homomorphisms

$$h^q : H^q(F, \mathbb{C}) \to H^q(F, \mathbb{C}) \;\; \forall\, 0 \leq q \leq 2.$$

We have the following decomposition

$$H^q(F, \mathbb{C}) = \bigoplus_{k=0}^{d} H^q(F)_{\lambda^k},$$

where $H^q(F)_{\lambda^k} = ker\{h^q - \lambda^k Id\}$ is the eigenspace associated to λ^k.

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where $\bar{\mathcal{L}}_{\lambda^k}$ is the rank one local system such that each monodromy around the hyperplanes of $\bar{\mathcal{A}}$ is λ^k.

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Problem

For $q = 1$:

$$H^1(F)_1 = H^1(M(\bar{A}), \mathbb{C}) \simeq \mathbb{C}^d$$

is completely determined by the intersection lattice.
The other eigenspaces $H^1(F)_{\lambda^k},\ \lambda^k \neq 1,$ are difficult to compute in general.

It has been known that the order of nontrivial eigenvalue $\lambda^k \neq 1$ is related to the multiplicities of points on a line $\bar{H}_j \in \bar{\mathcal{A}}$.

Definition

Let $q > 1$ be an integer larger than 1. Then $\mu(\bar{H}_j, q)$ is the number of points on $\bar{H}_j$ with multiplicities divisible by q.

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Theorem (Libgober)

Let $q > 1$ and $\lambda^k \neq 1$ be a non trivial eigenvalue of order q.

$$\text{If } \mu(\bar{H}_j, q) = 0 \text{ for some } \bar{H}_j \in \bar{\mathcal{A}}, \text{ then } H^1(F)_{\lambda^k} = 0.$$

Theorem (Yoshinaga)

Assume that $\bar{\mathcal{A}}$ is defined over $\mathbb{R}$.
Let $q > 1$ and $\lambda^k \neq 1$ be a non trivial eigenvalue of order q.

$$\text{If } \mu(\bar{H}_j, q) \leq 1 \text{ for some } \bar{H}_j \in \bar{\mathcal{A}}, \text{ then } H^1(F)_{\lambda^k} = 0.$$

Question

Can we generalize the above result when $\bar{\mathcal{A}}$ is defined over $\mathbb{C}$?

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Main result

Theorem (I)

Let $p \in \mathbb{Z}$ be a prime and $q = p^s$, $s \geq 1$ be its power satisfying $q|(d+1)$.
Let $\lambda^k \neq 1$ be a non trivial eigenvalue of order q.
Assume that $\bar{\mathcal{A}}$ has at least two intersections in $\mathbb{P}^2_{\mathbb{C}}$.

$$\text{If } \mu(\bar{H}_j, p) \leq 1 \text{ for some } \bar{H}_j \in \bar{\mathcal{A}}, \text{ then } H^1(F)_{\lambda^k} = 0.$$

Remark

Nontrivial eigenspaces can appear if $\mu(H_j, k) \geq 2$ for any line (see the $\mathcal{A}_3-$arrangement).

Proof

We consider $\mathcal{A} = \{H_1, \ldots, H_d\} \subset \mathbb{C}^2$ the deconing of $\bar{\mathcal{A}}$.

Let R be a commutative ring.

Let $A^*_R(\mathcal{A}) \simeq H^*(M(\mathcal{A}), R)$ be the Orlik-Solomon algebra of $\mathcal{A}$.

Let $\omega_1 = a_1 + a_2 + \ldots + a_d \in A^1_R(\mathcal{A})$, where $a_k = \frac{1}{2i\pi} \cdot \frac{d\alpha_k}{\alpha_k}$.

We consider the Aomoto complex:

$$(A^*_R(\mathcal{A}), \omega_1 \wedge) = \{\ A^*_R(\mathcal{A}) \xrightarrow{\omega_1 \wedge} A^{*+1}\ \}_{* \geq 0}.$$

The two following theorems imply our Theorem (I).

Proof

Theorem (Papadima, Suciu)

Let $p \in \mathbb{Z}$ be a prime, and $\lambda^k \neq 1$ be a nontrivial eigenvalue of order p^s, $s \geq 1$. Then

$$dim_{\mathbb{C}} H^1(M(\mathcal{A}), \mathcal{L}_{\lambda^k}) \leq dim_{\mathbb{F}_p} H^1(A^*_{\mathbb{F}_p}(\mathcal{A}), \omega_1 \wedge).$$

Theorem (II)

Let $p \in \mathbb{Z}$ be a prime.
Assume that $\bar{A}$ has at least two intersections in $\mathbb{P}^2_{\mathbb{C}}$.

$$\text{If } \mu(\bar{H}_0, p) \leq 1, \text{ then } H^1(A^*_{\mathbb{F}_p}(\mathcal{A}), \omega_1 \wedge) = 0.$$

If we suppose $\mu(\bar{H}_0, p) \leq 1$ these two theorems directly imply

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Proof

Theorem (I)

Let $p \in \mathbb{Z}$ be a prime and $q = p^s$, $s \geq 1$ be its power satisfying $q|(d+1)$.
Let $\lambda^k \neq 1$ be a non trivial eigenvalue of order q.
Assume that $\bar{\mathcal{A}}$ has at least two intersections in $\mathbb{P}^2_{\mathbb{C}}$.

$$\text{If } \mu(\bar{H}_i, p) \leq 1 \text{ for some } \bar{H}_i \in \bar{\mathcal{A}}, \text{ then } H^1(F)_{\lambda^k} = 0.$$

Proof

Difficulty: proof of Theorem (II)

Theorem (II)

Let $p \in \mathbb{Z}$ be a prime.
Assume that $\bar{A}$ has at least two intersections in $\mathbb{P}^2_{\mathbb{C}}$.

$$\text{If } \mu(\bar{H}_0, p) \leq 1, \text{ then } H^1(A^*_{\mathbb{F}_p}(\mathcal{A}), \omega_1 \wedge) = 0.$$

Idea: Use the topology of $M(\mathcal{A})$.
Since $A^*_R(\mathcal{A})$ can be very complicated, we can reduced by projection to simpler cases (central arrangement, almost parallel lines).

Zeta fonction

Why can we only consider $H^1(F,\mathbb{C})$?

Because of the zeta fonction, which allows us to obtain $H^2(F,\mathbb{C})$ from $H^1(F,\mathbb{C})$.

Definition

Let $\bar{A} \subset \mathbb{P}^2$ be an hyperplane arrangement. The Zeta fonction is the polynomial

$$Z(h)(t) = \frac{\Delta^0(t).\Delta^2(t)}{\Delta^1(t)},$$

where the $\Delta^q(t) = det(t.Id_{H^q(F,\mathbb{C})} - h^q)$ are the Alexander polynomials.

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Zeta fonction

If $|\bar{\mathcal{A}}| = d + 1$, we can show that

$$\Delta^q(t) = \prod_{k=0}^{d} (t - \lambda^k)^{dimH^q(F)_{\lambda^k}},$$

and

$$Z(h)(t) = \prod_{k=0}^{d} (t - \lambda^k)^{E(M(\bar{\mathcal{A}}))},$$

where $E(M(\bar{\mathcal{A}}))$ is the Euler characteristic of $M(\bar{\mathcal{A}}),$ determined by the intersection lattice of $\bar{\mathcal{A}}$.
Hence we have

$$\Delta^2(t) = \frac{\Delta^1(t). \prod_{k=0}^{d}(t - \lambda^k)^{E(M(\bar{\mathcal{A}}))}}{(t-1)}.$$

Zeta fonction

If the assumptions of our Theorem (I) are satisfied, then $H^1(F,\mathbb{C}) = H^1(F)_1$ and we have:

$$\Delta^2(t) = (t-1)^{d-2}\prod_{k=0}^{d}(t-\lambda^k)^{E(M(\bar{\mathcal{A}}))},$$

so

$$dim\, H^2(F)_1 = E(M(\bar{\mathcal{A}})) + d - 2$$

$$dim\, H^2(F)_{\lambda^k} = E(M(\bar{\mathcal{A}}))\ \ \forall\, \lambda^k \neq 1.$$

Total degeneration

Let $\mathcal{A} = \{H_1, \ldots, H_d\} \subset \mathbb{C}^2$ be the deconing of $\mathcal{A}$ and its partition in t parallel classes:

$$\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \ldots \sqcup \mathcal{A}_t.$$

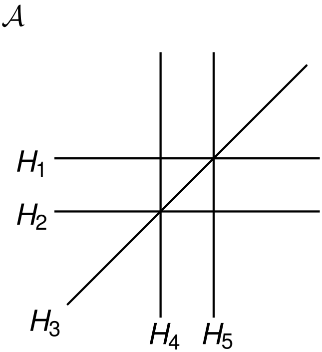
Theorem

We have a surjective homomorphism:

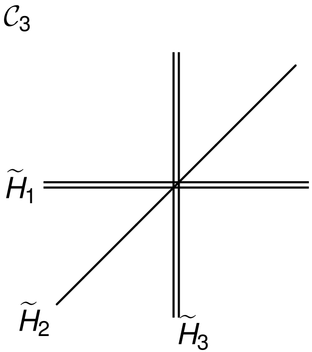
$$\Delta_{tot} : A_R^*(\mathcal{A}) \longrightarrow A_R^*(\mathcal{C}_t),$$

where $\mathcal{C}_t$ is a central arrangement of t lines in $\mathbb{C}^2$.

Example I



$\mathcal{A}_1 = \{H_1, H_2\}$
$\mathcal{A}_2 = \{H_3\}$
$\mathcal{A}_3 = \{H_4, H_5\}$



Total degeneration of $\mathcal{A}$ in $\mathcal{C}_3$.

Directional degeneration

Let $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \ \dots \ \sqcup \mathcal{A}_t$ be the decomposition of our arrangement.

Choose a parallel class $\mathcal{A}_\beta$.

Let $v = |\mathcal{A}_\beta|$.

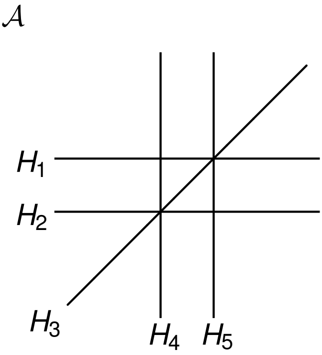
Theorem

We have a surjective homomorphism, called directional degeneration with respect to $\mathcal{A}_\beta$

$$\Delta_{dir} : A^*_R(\mathcal{A}) \longrightarrow A^*_R(\mathcal{P}_v),$$

where $\mathcal{P}_v$ is composed of v parallel lines and an other line transversal to them.

Example II



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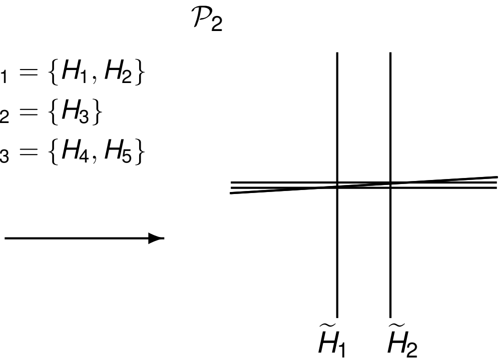

Directional degeneration of $\mathcal{A}$ to $\mathcal{P}_2$ with respect to $\mathcal{A}_3$.

Thanks to the degeneration homomorphisms,
we can reduced by projection to simpler Orlik-Solomon algebras and proove Theorem (II).

Thank you for your attention...