Generalization of the degeneration homomorphisms in higher dimensions

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Let $\mathcal{A} = \{H_1, \ldots, H_n\} \subset \mathbb{C}^l$ be an affine hyperplane arrangement, and $\bar{\mathcal{A}} := \{\overline{H}_1, \ldots, \overline{H}_n, H_\infty\} \subset \mathbb{P}^l_{\mathbb{C}}$ the coning of $\mathcal{A}$, obtained by adding the hyperplane at infinity $H_\infty := \{x_{l+1} = 0\}$. Equivalently, $\bar{\mathcal{A}}$ is a central arrangement in $\mathbb{C}^{l+1}$ and $\mathcal{A} = d_{H_\infty}\bar{\mathcal{A}}$ is the deconing of $\mathcal{A}$ with respect to H_∞. Let us note $\overline{\alpha}_i$ the linear form defining each hyperplane $\overline{H}_i$, i.e. $\overline{H}_i := \{(x_1, \ldots, x_{l+1}) \in \mathbb{C}^{l+1} \,|\, \overline{\alpha}_i(x_1, \ldots, x_{l+1}) = \sum_{j=1}^{l+1} a_{j,i}x_j = 0\}$. Note that $\alpha_i(x_1, \ldots, x_l) = \overline{\alpha}_i(x_1, \ldots, x_l, 1)$. For any $X \in L(\bar{\mathcal{A}})$, $X \subset H_\infty$, we denote by $\bar{\mathcal{A}}_X = \{\overline{H}_i \in \bar{\mathcal{A}} \,|\, X \subset \overline{H}_i\} \cup H_\infty$ the corresponding subarrangement. Note that $\bar{\mathcal{A}}_X$ has rank $\operatorname{codim}(X)$ and is not essential as soon as $\operatorname{codim}(X) \geq 1$. Let us note $r = \operatorname{codim}(X)$. We denote by $\frac{\bar{\mathcal{A}}_X}{X}$ the corresponding essential arrangement in $\mathbb{C}^r$. Let us consider two arrangements:

- $d_{H_\infty}(\frac{\bar{\mathcal{A}}_X}{X}) \subset \mathbb{C}^{r-1}$, the deconing of $\frac{\bar{\mathcal{A}}_X}{X}$ with respect to H_∞
- $\bar{\mathcal{A}}^X = \{\overline{H}_i \cap X, \forall \overline{H}_i \in \bar{\mathcal{A}} \backslash \bar{\mathcal{A}}_X \text{ s.t. } \overline{H}_i \cap X \neq \emptyset\} \subset X \simeq \mathbb{C}^{l+1-r}$.

With these notations we have the following definition.

Definition 0.1. The *degenerated arrangement* of $\mathcal{A}$ with respect to X is

$$\widetilde{\mathcal{A}} = d_{H_\infty}(\frac{\bar{\mathcal{A}}_X}{X}) \times \bar{\mathcal{A}}^X \subset \mathbb{C}^l.$$

And we can define a degeneration homomorphism of Orlik-Solomon algebras that generalize in any dimension the degeneration homomorphisms introduced in [2].

Theorem 0.2. *Let R be a commutative ring. There exists a surjective homomorphism, called degeneration homomorphism with respect to X*

$$\Delta_X : A^*_R(\mathcal{A}) \twoheadrightarrow A^*_R(d_{H_\infty}(\frac{\bar{\mathcal{A}}_X}{X})) \otimes A^*_R(\bar{\mathcal{A}}^X).$$

Proof. Let $e_i \in A^1_R(\mathcal{A})$ be the generator associated to the hyperplane $H_i \in \mathcal{A}$. Assume X is a codimension r edge of $\bar{\mathcal{A}}$ contained in H_∞, and $\bar{\mathcal{A}}_X = \{\overline{H}_1, \ldots, \overline{H}_k\} \cup H_\infty$. Then there exist r hyperplanes linearly independent in $\bar{\mathcal{A}}_X$ whose intersection is X. For simplicity we will assume $X = \overline{H}_1 \cap \ldots \cap \overline{H}_{r-1} \cap H_\infty$, and $\overline{\alpha}_i(x_1, \ldots, x_{l+1}) = x_i,\ \forall\, 1 \leq i \leq r-1$. In other words, $X = \{x_1 = \cdots = x_{r-1} = x_{l+1} = 0\}$. Note that for any i, $r \leq i \leq k$, the linear forms $\overline{\alpha}_i$ satisfy $a_{j,i} = 0, \forall\, r \leq j \leq l$. We have an isomorphism $\mathbb{C}^{l+1} \simeq X \oplus E$, where $E = \{(x_1, \ldots, x_{l+1}) \in \mathbb{C}^{l+1} \,|\, x_r = \cdots = x_l = 0\} \simeq \mathbb{C}^r$. The essential arrangement $\frac{\bar{\mathcal{A}}_X}{X} = \{\frac{\overline{H}_1}{X}, \ldots, \frac{\overline{H}_k}{X}\} \cup \frac{\overline{H}_\infty}{X}$ is obtained by projecting the hyperplanes of $\bar{\mathcal{A}}_X$ on E, i.e. $\frac{\overline{H}_i}{X} := \{(x_1, \ldots, x_{r-1}, x_{l+1}) \in \mathbb{C}^r \mid \sum_{j=1}^{r-1} a_{j,i}x_j + a_{l+1,i}x_{l+1} = 0\}$. Hence the hyperplanes of $d_{H_\infty}(\frac{\bar{\mathcal{A}}_X}{X})$ are defined by $d_{H_\infty}(\frac{\overline{H}_i}{X}) = \frac{H_i}{X} = \{(x_1, \ldots, x_{r-1}) \in \mathbb{C}^{r-1} \mid \sum_{j=1}^{r-1} a_{j,i}x_j + a_{l+1,i} = 0\}$. Finally, the arrangement $\widetilde{\mathcal{A}}_1 = d_{H_\infty}(\frac{\bar{\mathcal{A}}_X}{X}) \times \mathbb{C}^{l-r+1} = \{\widetilde{H}_1, \ldots, \widetilde{H}_k\} \subset \mathbb{C}^l$ has hyperplanes $\widetilde{H}_i = \{(x_1, \ldots, x_l) \in \mathbb{C}^l \mid \sum_{j=1}^{r-1} a_{j,i}x_j + a_{l+1,i} = 0\}$. Let us note $\widetilde{\alpha}_i(x_1, \ldots, x_l) = \sum_{j=1}^{r-1} a_{j,i}x_j + a_{l+1,i}$ the defining equation of $\widetilde{H}_i$, for any $1 \leq i \leq k$, and $\widetilde{e}_i \simeq \frac{1}{2\sqrt{-1}\pi} d\log(\widetilde{\alpha}_i) \in A^*_R(\widetilde{\mathcal{A}}_1)$ the generator of the Orlik-Solomon algebra of $\widetilde{\mathcal{A}}_1$ associated to the hyperplane $\widetilde{H}_i$.
Any hyperplane $\overline{H}_i \in \bar{\mathcal{A}} \backslash \bar{\mathcal{A}}_X$, i.e. with $k+1 \leq i \leq n$, has a defining equation $\overline{\alpha}_i(x_1, \ldots, x_{l+1}) = \sum_{j=1}^{l+1} a_{j,i}x_j$ satisfying $(a_{r,i}, \ldots, a_{l,i}) \neq (0, \ldots, 0)$ and each intersection $\overline{H}_i \cap X \subset X \simeq \mathbb{C}^{l+1-r}$ is defined by the equation $\sum_{j=r}^{l} a_{j,i}x_j = 0$. Note that the $\overline{H}_i \cap X$ are not necessarily distinct. Finally, the arrangement $\widetilde{\mathcal{A}}_2 = \mathbb{C}^{r-1} \times \bar{\mathcal{A}}^X = \{\widetilde{H}_{k+1}, \ldots, \widetilde{H}_n\} \subset \mathbb{C}^l$ has hyperplanes $\widetilde{H}_i = \{(x_1, \ldots, x_l) \in \mathbb{C}^l \mid \sum_{j=r}^{l} a_{j,i}x_j = 0\}$. Let us note $\widetilde{\alpha}_i(x_1, \ldots, x_l) = \sum_{j=r}^{l} a_{j,i}x_j$ the defining equation of $\widetilde{H}_i$, for any $k+1 \leq i \leq n$, and $\widetilde{e}_i \simeq \frac{1}{2\sqrt{-1}\pi} d\log(\widetilde{\alpha}_i) \in A^*_R(\widetilde{\mathcal{A}}_2)$ the generator of the Orlik-Solomon algebra of $\widetilde{\mathcal{A}}_2$ associated to the hyperplane $\widetilde{H}_i$.
Clearly, the degenerated arrangement $d_{H_\infty}(\frac{\bar{\mathcal{A}}_X}{X}) \times \bar{\mathcal{A}}^X$ is $\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2$, and there is a surjective homomorphism of $R-$algebras $f : E^*_R(\mathcal{A}) \longrightarrow A^*_R(\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2)$ such that $f(e_i) = \widetilde{e}_i$, for all $1 \leq i \leq n$. Let us show that $I(\mathcal{A}) \subset \ker(f)$.

1. Let $y = e_{i_1} \wedge \cdots \wedge e_{i_s} \in I(\mathcal{A})$, such that $H_{i_1} \cap \cdots \cap H_{i_s} = \emptyset$. If $\widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_s} = \emptyset$, then $f(y) = 0$. Assume now that $\widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_s} \neq \emptyset$. We directly see that there exists at least one hyperplane among $\widetilde{H}_{i_1}, \ldots, \widetilde{H}_{i_s}$ which is in $\widetilde{\mathcal{A}}_2$. Otherwise we would have $H_{i_1} \cap \cdots \cap H_{i_s} = \emptyset \Rightarrow \widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_s} = \emptyset$. So let p, $0 \leq p \leq s-1$, be such that $\widetilde{H}_{i_1}, \ldots, \widetilde{H}_{i_p} \in \widetilde{\mathcal{A}}_1$ and $\widetilde{H}_{i_{p+1}}, \ldots, \widetilde{H}_{i_s} \in \widetilde{\mathcal{A}}_2$. Let us show that $\widetilde{H}_{i_1}, \ldots, \widetilde{H}_{i_s}$ are not linearly independent. If $\widetilde{H}_{i_1}, \ldots, \widetilde{H}_{i_s}$ were linearly independent, we could choose

the coordinates such that $\overline{H}_{i_{p+t}} \cap X := \{x_{r-1+t} = 0\}$, $\forall\, 1 \leq t \leq s$. Hence we would have that $\alpha_{i_{p+t}}(x_1, \ldots, x_l) = \sum_{j=1}^{r-1} a_{j,i_{p+t}} x_j + x_{r-1+t} + a_{l+1,i_{p+t}}$. Furthermore, the defining equations of $\overline{H}_{i_t}$, for $1 \leq t \leq p$ are on the form $\overline{\alpha}_{i_t}(x_1, \ldots, x_{l+1}) = \sum_{j=1}^{r-1} a_{j,i_t} x_j + a_{l+1,i_t} x_{l+1}$. It follows that $\alpha_{i_t}(x_1, \ldots, x_l) = \sum_{j=1}^{r-1} a_{j,i_t} x_j + a_{l+1,i_t}$ for all $1 \leq t \leq p$. We easily see that $H_{i_1} \cap \cdots \cap H_{i_s} = \emptyset \Rightarrow H_{i_1} \cap \cdots \cap H_{i_p} = \emptyset \Rightarrow \widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_p} = \emptyset \Rightarrow \widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_s} = \emptyset$, which contradicts our assumption.

2. Let $y = \partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) \in I(\mathcal{A})$, such that $H_{i_1} \cap \cdots \cap H_{i_s} \neq \emptyset$ and $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_s}) < s$. If $\widetilde{H}_{i_1}, \ldots, \widetilde{H}_{i_s} \in \widetilde{\mathcal{A}}_1$, then we easily see that $\widetilde{H}_{i_1}, \ldots, \widetilde{H}_{i_s}$ have non empty intersection of codimension $< s$. Assume now that $\widetilde{H}_{i_1}, \ldots, \widetilde{H}_{i_p} \in \widetilde{\mathcal{A}}_1$ and $\widetilde{H}_{i_{p+1}}, \ldots, \widetilde{H}_{i_s} \in \widetilde{\mathcal{A}}_2$, for a certain $0 \leq p \leq s-1$. Let us note $q = \operatorname{codim}(\widetilde{H}_{i_{p+1}} \cap \ldots \cap \widetilde{H}_{i_s})$. We have that $H_{i_1} \cap \cdots \cap H_{i_s} \neq \emptyset \Rightarrow H_{i_1} \cap \cdots \cap H_{i_p} \neq \emptyset \Rightarrow \widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_p} \neq \emptyset \Rightarrow \widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_s} \neq \emptyset$, since $\widetilde{\mathcal{A}}_2$ is central and the defining polynomials involve different variables for hyperplanes in $\widetilde{\mathcal{A}}_2$ or $\widetilde{\mathcal{A}}_1$. Let us show that $\operatorname{codim}(\widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_s}) < s$. If $q < s-p$, then $\operatorname{codim}(\widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_s}) \leq \operatorname{codim}(\widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_p}) + q \leq p + q < s$. If $q = s-p$, then $\widetilde{H}_{i_{p+1}}, \ldots, \widetilde{H}_{i_s}$ are linearly independent and we can choose the coordinates such that $\widetilde{\alpha}_{i_{p+t}} = x_{r-1+t}$, for any $1 \leq t \leq s-p$. It is easy to check that in this case $\operatorname{codim}(\widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_s}) = \operatorname{codim}(\widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_p}) + s - p = \operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_p}) + s - p$, and that $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_s}) = \operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_p}) + s - p$. Finally, $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_p}) = \operatorname{codim}(\widetilde{H}_{i_1} \cap \cdots \cap \widetilde{H}_{i_p}) < p \Rightarrow \widetilde{H}_{i_1}, \ldots, \widetilde{H}_{i_s}$ are not linearly independent.

Since $I(\mathcal{A}) \subset \ker(f)$ we deduce with the universal property of algebras the existence of the degeneration homomorphism Δ_X. □

Let us consider a decomposition of $\mathcal{A}$ into classes

$$\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_s,$$

such that two hyperplanes $H_i, H_j \in \mathcal{A}$ belong to a same class $\mathcal{A}_\beta$, $1 \leq \beta \leq s$, if and only if $\overline{H}_i \cap H_\infty = \overline{H}_j \cap H_\infty \Leftrightarrow H_i \cap H_j = \emptyset$. Note that to any class $\mathcal{A}_\beta$ corresponds the codimension 2 edge $X = \bigcap_{i \,|\, H_i \in \mathcal{A}_\beta} \overline{H}_i \cap H_\infty$, and that X is dense if and only if $|\mathcal{A}_\beta| \geq 2$.

Let $\lambda = (\lambda_i)_{i \in \{1,\ldots,n\}} \in \mathbb{C}^n$ and $\omega_\lambda = \sum_{i=1}^{n} \lambda_i e_i \in A_R^1(\mathcal{A})$. As in the previous section, we will study the Aomoto complex $(A_R^*(\mathcal{A}), \omega_\lambda \wedge) = \{A_R^*(\mathcal{A}) \xrightarrow{\omega_\lambda \wedge}$

$A_R^{*+1}(\mathcal{A})\}$ and show that its first cohomology group vanishes under certain non resonance condition.

Theorem 0.3. *Assume* $\lambda_\infty = -\sum_{i=1}^n \lambda_i \in R^\times$ *and* $\lambda_X = \sum_{i\,|\,X \not\subseteq \overline{H}_i} \lambda_i \in R^\times$, *for all* $X \in D_2(\bar{\mathcal{A}})$, $X \subseteq H_\infty$. *Then*

$$H^1(A_R^*(\mathcal{A}), \omega_\lambda \wedge) = 0.$$

Proof. In order to prove this theorem, we need the three following lemmas.

Lemma 0.4. *([2, Lemma 1]) Let* $\mathcal{A} = \{H_1, \ldots, H_n\} \subset \mathbb{C}^l$ *be an affine hyperplane arrangement. Define the submodule* $A_R^1(\mathcal{A})_0$ *of* $A_R^1(\mathcal{A})$ *by*

$$A_R^1(\mathcal{A})_0 := \{\eta = c_1e_1 + \cdots + c_ne_n \in A_R^1(\mathcal{A}) \mid c_1 + c_2 + \cdots + c_n = 0\}.$$

Let $\xi = a_1e_1 + \cdots + a_ne_n \in A_R^1(\mathcal{A})$. *If* $\sum_{i=1}^n a_i \in R^\times$, *then*

$$H^1(A_R^*(\mathcal{A}), \xi\wedge) \simeq \ker\left(A_R^1(\mathcal{A})_0 \xrightarrow{\xi\wedge} A_R^2(\mathcal{A})\right).$$

Lemma 0.5. *([2, Proposition]) Let* $\mathcal{B} = \{H_1, \ldots, H_s\} \subset \mathbb{C}^l$ *be a* ***central*** *affine hyperplane arrangement and* $\xi = a_1e_1 + \cdots + a_se_s \in A_R^1(\mathcal{B})$. *If* $\sum_{i=1}^s a_i \in R^\times$, *then* $H^k(A_R^*(\mathcal{B}), \xi\wedge) = 0$, *for all* $k \geq 1$.

Proof. Let $\eta \in \ker\{A_R^k(\mathcal{B}) \xrightarrow{\xi\wedge} A_R^{k+1}(\mathcal{B})\}$. Then $\xi \wedge \eta = 0 \Rightarrow \partial(\xi \wedge \eta) = (\sum_{i=1}^s a_i)\eta - \xi \wedge \partial(\eta) = 0 \Rightarrow \eta = (\sum_{i=1}^s a_i)^{-1}\xi \wedge \partial(\eta) \in \mathrm{Im}\{A_R^{k-1}(\mathcal{B}) \xrightarrow{\xi\wedge} A_R^k(\mathcal{B})\}$. □

Lemma 0.6. *Let* $X \in D_2(\bar{\mathcal{A}})$, $X \subseteq H_\infty$. *Then* $\bar{\mathcal{A}}^X = \{\overline{H}_i \,|\, H_i \in \mathcal{A}_\beta\} \cup H_\infty$, *for a certain* $1 \leq \beta \leq s$. *We have that* $\widetilde{e}_i = \widetilde{e}_j$, *for all* H_i, H_j *that belong to a same class* $\mathcal{A}_\alpha$ *with* $\alpha \neq \beta$.

Proof. By definition of the class $\mathcal{A}_\alpha$, if H_i and H_j belong to $\mathcal{A}_\alpha$ then $\overline{H}_i \cap H_\infty = \overline{H}_j \cap H_\infty$. It follows that $\overline{H}_i \cap H_\infty \cap X = \overline{H}_j \cap H_\infty \cap X \Rightarrow \overline{H}_i \cap X = \overline{H}_j \cap X \Rightarrow \widetilde{e}_i = \widetilde{e}_j$, since $X \subseteq H_\infty$. □

We are now ready to prove Theorem 0.3. For simplicity we order the hyperplanes of $\mathcal{A}$ such that $\mathcal{A}_1 = \{H_1 \ldots, H_{i_1}\}, \mathcal{A}_2 = \{H_{i_1+1}, \ldots, H_{i_2}\}, \ldots, \mathcal{A}_s = \{H_{i_{s-1}+1}, \ldots, H_n\}$. Let $\eta \in H^1(A_R^*(\mathcal{A}), \omega_\lambda\wedge)$. It is easy to check that $A_R^1(\mathcal{A})_0$ is generated by the elements $e_i - e_{i+1}$, $1 \leq i \leq n-1$, and with Lemma 0.4 we can write $\eta = \sum_{i=1}^{n-1} c_i(e_i - e_{i+1}) \in A_R^1(\mathcal{A})_0$, with $\omega_\lambda \wedge \eta = 0$. By taking $X = H_\infty$, the degenerated arrangement $\widetilde{\mathcal{A}}$ satisfies $\widetilde{\mathcal{A}}_1 = \emptyset$ and

$\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_2 = \bar{\mathcal{A}}^{H_\infty}$ is central. With the property of the classes appearing in the decomposition of $\mathcal{A}$, we compute easily that

$$\Delta_X(\eta) = \sum_{\beta=1}^{s-1} c_{i_\beta}(\widetilde{e}_\beta - \widetilde{e}_{\beta+1}) \in A^1_R(\widetilde{\mathcal{A}})_0 \tag{1}$$

and

$$\Delta_X(\omega_\lambda) = \sum_{\beta=1}^{s} (\sum_{i \,|\, H_i \in \mathcal{A}_\beta} \lambda_i)\widetilde{e}_\beta. \tag{2}$$

Furthermore, we have that $\Delta_X(\omega_\lambda \wedge \eta) = \Delta_X(\omega_\lambda) \wedge \Delta_X(\eta) = 0$ and that $\Delta_X(\omega_\lambda)$ and $\Delta_X(\eta)$ satisfy the assumptions of Lemmas 0.4 and 0.5. We deduce directly that $\eta \in \ker(\Delta_X)$, i.e $c_{i_\beta} = 0$ for all β, $1 \leq \beta \leq s-1$. Hence

$$\eta = \sum_{\substack{1 \leq \beta \leq s \text{ s.t.} \\ |\mathcal{A}_\beta| \geq 2}} \; \sum_{\substack{i \text{ s.t.} \\ H_i, H_{i+1} \in \mathcal{A}_\beta}} c_i(e_i - e_{i+1}). \tag{3}$$

Let us now consider a dense edge $X \in D_2(\bar{\mathcal{A}})$, $X \subseteq H_\infty$. Then $\bar{\mathcal{A}}^X = \{\overline{H}_i \,|\, H_i \in \mathcal{A}_\beta\} \cup H_\infty$, for a certain $1 \leq \beta \leq s$. For simplicity, we will assume that $\beta = 1$. With Lemma 0.6, we compute easily

$$\Delta_X(\eta) = \sum_{i=1}^{i_1-1} c_i(\widetilde{e}_i - \widetilde{e}_{i+1}) \in A^1_R(\widetilde{\mathcal{A}})_0 \tag{4}$$

and

$$\Delta_X(\omega_\lambda) = \widetilde{\omega_\lambda} = \widetilde{\omega_{\lambda 1}} + \widetilde{\omega_{\lambda 2}}, \tag{5}$$

where $\widetilde{\omega_{\lambda 1}} = \sum_{i=1}^{i_1} \lambda_i \widetilde{e}_i \in A^1_R(\widetilde{\mathcal{A}}_1)$ and $\widetilde{\omega_{\lambda 2}} = \sum_{\beta=2}^{s} (\sum_{i \,|\, H_i \in \mathcal{A}_\beta} \lambda_i)\widetilde{e}_{i_1+\beta-1} \in A^1_R(\widetilde{\mathcal{A}}_2)$.

Note that the $\widetilde{e}_{i_1+\beta-1}$ are not necessarily distinct. We have that $\partial(\widetilde{\omega_{\lambda 2}}) = \lambda_X \in R^\times$ by assumption and with Lemma 0.5 it follows $H^k(A^*_R(\widetilde{\mathcal{A}}_2), \widetilde{\omega_{\lambda 2}} \wedge) = 0$, $\forall\, k \geq 0$ since $\widetilde{\mathcal{A}}_2$ is central. Hence

$$H^1(A^*_R(\widetilde{\mathcal{A}}), \widetilde{\omega_\lambda} \wedge) = \bigoplus_{s+t=1} H^s(A^*_R(\widetilde{\mathcal{A}}_1), \widetilde{\omega_{\lambda 1}} \wedge) \otimes H^t(A^*_R(\widetilde{\mathcal{A}}_2), \widetilde{\omega_{\lambda 2}} \wedge) = 0.$$

Furthermore, we have that $\Delta_X(\omega_\lambda \wedge \eta) = \Delta_X(\omega_\lambda) \wedge \Delta_X(\eta) = 0$ and that $\Delta_X(\omega_\lambda)$ and $\Delta_X(\eta)$ satisfy the assumptions of Lemmas 0.4. We deduce directly that $\eta \in \ker(\Delta_X)$, i.e $c_i = 0$ for all i, $1 \leq i \leq i_1$.

By applying the same method to all the classes β such that $|\mathcal{A}_\beta| \geq 2$, we find $\eta = 0$. □

The authors think that the degeneration homomorphism Δ_X could be applied to generalize Theorem 0.3 to the higher cohomology groups.

Conjecture 0.7. *If* $\lambda_\infty \in R^\times$ *and* $\lambda_X \in R^\times$ *for any dense edge* $X \in D_{\leq q}(\bar{\mathcal{A}})$ *with* $X \subset H_\infty$*, then* $H^k(A^*_R(\mathcal{A}), \omega_\lambda \wedge) = 0$, $\forall 1 \leq k \leq q-1$.

Remark 0.8. Conjecture 0.7 has been recently proved in [1] for real essential hyperplane arrangements $\mathcal{A} \subset \mathbb{R}^l$ by using minimality of hyperplane arrangements.

References

[1] P. Bailet, M. Yoshinaga: Vanishing results for the Aomoto complex of real hyperplane arrangements via minimality; arXiv:1512.05318

[2] Pauline Bailet, Masahiko Yoshinaga: Degeneration of Orlik-Solomon algebras and Milnor fibers of complex line arrangements; Geometriae Dedicata, 2014, 10.1007/s10711-014-0027-7