A vanishing result for the first twisted cohomology of affine varieties and applications to line arrangements

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Fact : Recall that U is affine if and only if it exists an effective ample divisor $\widetilde{D} = \sum_{i=1}^{n} a_i D_i$ ($a_i \in \mathbb{Q}_{>0}$).

Proposition (Nakai-Moishezon criterion)

Let L be a line bundle on S. Then L is ample if and only if $L^2 > 0$ and $L \cdot C > 0$ for all irreducible curves $C \subset S$.

Idea : Apply the Nakai-Moishezon criterion to the line bundle $L = \mathcal{O}(\widetilde{D})$.

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Remarks :

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Remarks :

- it is sometimes possible to consider subdivisors $D' = \sum_{i \in I} a_i D_i$ of D instead of D.
- We need to check the existence of "bad curves" $C \subset S$ such that $C \cap D = \emptyset$ (since in this case $\widetilde{D} \cdot C = 0$ and U is not affine).

Let S be a smooth proper complex variety of dimension ≥ 2, $D = \sum_{i=1}^{n} D_i \subset S$ a reduced divisor and $U = S \setminus D$ the corresponding complement.

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Aim : Vanishing result for $H^1(U, \mathcal{L})$?

Consider a subdivisor of D :

$$D' = \sum_{i \in I} D_i,$$

where $I \subset [n] = \{1, \ldots, n\}$.

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Theorem (Dimca, Yoshinaga, B.)

Assume that

(i) *D is normal crossing along D', that is, for any $p \in D'$, D is normal crossing around p.*

(ii) *$t_i \neq 1$ for $i \in I$.*

(iii) *$U' = S \smallsetminus D'$ is an affine variety.*

Then $H^1(U, \mathcal{L}) = 0$.

Let $\mathcal{A} = \{H_1, \ldots, H_n\} \subset \mathbb{P}^2$ be a line arrangement.

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Let $\mathcal{L}$ be a rank one local system on the complement

$$M(\mathcal{A}) = \mathbb{P}^2 \setminus \bigcup_{i=1}^{n} H_i,$$

with monodromy $t_i \in \mathbb{C}^\times$ around each H_i.
Assume $t_i \neq 1$ for all $i = 1, \ldots, n$.

Of course, $\bigcup_{i=1}^{n} H_i$ is not a normal crossing divisor !

Of course, $\bigcup_{i=1}^{n} H_i$ is not a normal crossing divisor !

Idea : Do some blowings-ups, find good candidate for the subdivisor D' such that U' is affine and apply our main result.

Given a point $p \in \mathbb{P}^2$, we denote by

$$t_p = \prod_{H_i \ni p} t_i$$

the corresponding total turn monodromy of $\mathcal{L}$.

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Let $T := \{p \in \mathbb{P}^2 \mid mult(p) \geq 3\}$. The set T is decomposed into

$$T = T_{\neq 1} \sqcup T_{=1},$$

where $T_{\neq 1} := \{p \in T \mid t_p \neq 1\}$ are the "good points for $\mathcal{L}$" and $T_{=1} := \{p \in T \mid t_p = 1\}$ the "bad points".

Consider $B\ell_T\,\mathbb{P}^2$ the surface obtained by blowing up the points in T.

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Denote by E_p the exceptional divisor for $p \in T$ and by $\overline{H}_i$ the strict transform of the line H_i.

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Remark : The affinity of the complement is a difficult question and depends of the positions of the singularities !

Example

Let $p_1, \ldots, p_\ell \in \mathbb{P}^2$ be distinct points on a line L in $\mathbb{P}^2$.
Let $\mathcal{A}_i = \{H_{i,1}, \ldots, H_{i,c_i}\}$ be a set of lines passing through p_i and $\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_\ell$. We assume that $L \notin \mathcal{A}_i$ and all intersections of $\mathcal{A}$ except for $p_1, \ldots, p_\ell$ are double points. Suppose $t_{p_1} = \cdots = t_{p_\ell} = 1$.

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Consider

$$S = B\ell_T\,\mathbb{P}^2,\ D = \sum_{i=1}^{\ell}\sum_{j=1}^{c_i} \overline{H}_{i,j}.$$

Then the strict transform $\overline{L}$ does not intersect D, $D \cdot L = 0$, and hence $S \smallsetminus D$ is not affine.
Conclusion : When $p_1, \ldots, p_\ell \in T_{=1}$ are colinear, $S \smallsetminus D$ is not affine.

Example

Let $p_1, \ldots, p_6 \in \mathbb{P}^2$ be 6 points such that **no three are colinear**.
Let $\mathcal{A} = \{H_1, \ldots, H_9\}$ be the edges of the corresponding hexagon and three diagonals, such that each line H_j contains exactly 2 triple points and 4 nodes.
Assume $t_{p_i} = 1$ at all the vertices of the hexagon.
Consider

$$S = B\ell_T \mathbb{P}^2, \; D = \sum_{i=1}^{9} \overline{H}_i.$$

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The ampleness of D depends on the position of the six points $p_1, \ldots, p_6$:
if $p_1, \ldots, p_6$ are lying on a conic C, then $\overline{C} \cap D = \emptyset$ and $D \cdot \overline{C} = 0$.
Conclusion : When $p_1, \ldots, p_6 \in T_{=1}$ are lying on a conic, $S \setminus D$ is not affine.

Conclusion : We need to control the existence of "bad curves" as in the previous examples. This can be done by considerations on the degree of the curve, the multiplicity of the curve at the intersection points in $T_{=1}$, and the multiplicity of the points in the arrangement.

Some applications to our main Theorem :

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D. Cohen, A. Dimca, P. Orlik, Nonresonance conditions for arrangements. *Ann. Institut Fourier (Grenoble)*, **53** (2003), 1883–1896.

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Example (Cohen-Dimca-Orlik)

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ and $\mathcal{L}$ a rank one local system on the complement $M(\mathcal{A})$. Suppose there exists a line $H_k \in \mathcal{A}$ such that $H_k \cap T_{=1} = \emptyset$. Then $H^1(M(\mathcal{A}), \mathcal{L}) = 0$.

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Take :

- $S = B\ell_{T \cap H_k} \mathbb{P}^2$
- $D = \sum_{i=1}^{n} \overline{H}_i + \sum_{p \in T \cap H_k} E_p$
- $D' = \overline{H}_k + \sum_{p \in T \cap H_k} E_p.$

Remarks :

- This example follows from Libgober :

  A. Libgober, Eigenvalues for the monodromy of the Milnor fibers of arrangements. In : Libgober, A., Tibăr, M. (eds) *Trends in Mathematics : Trends in Singularities*. Birkhäuser, Basel (2002)

- The result of Cohen-Dimca-Orlik is a generalization of Libgober's result.

Proposition

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ and $\mathcal{L}$ a rank one local system on the complement $M(\mathcal{A})$. Suppose there exists a line $H_k \in \mathcal{A}$ such that

$$H_k \cap T_{=1} = \{p\},$$

and assume that there is no line L in $\mathbb{P}^2$, passing through the point p and such that $L \cap \mathcal{A} \subset T_{=1}$. Then $H^1(M(\mathcal{A}), \mathcal{L}) = 0$.

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Take :

- $S = B\ell_T\, \mathbb{P}^2$
- $D = \sum_{i=1}^n \overline{H}_i + \sum_{p \in T} E_p$
- $D' = \overline{H}_k + \sum_{i \,|\, p \notin H_k} \overline{H}_i + \sum_{q \in T_{\neq 1}} E_q$.

Note that this result was known for real line arrangements without the extra condition on the line L :

M. Yoshinaga, Milnor fibers of real line arrangements. *Journal of Singularities*, **7** (2013), 220-237.

M. Yoshinaga, Resonant bands and local system cohomology groups for real line arrangements. *Vietnam Journal of Mathematics*, **42**, 3, (2014) 377-392.

Milnor fiber

The **Milnor fiber** of $\mathcal{A}$ is the smooth affine hypersurface defined by

$$F = Q_{\mathcal{A}}^{-1}(1) \subset \mathbb{C}^3,$$

where

$$Q_{\mathcal{A}}(x, y, z) = \prod_{i=1}^{n} \alpha_i(x, y, z),$$

with $H_i := \{\alpha_i = 0\}$.

Monodromy

Let $\lambda = exp(2i\pi/n)$, $n = |\mathcal{A}|$. The **monodromy** on F is :

$$\begin{aligned} h: \quad & F \to F \\ & x \mapsto \lambda \cdot x \end{aligned}$$

We denote by

$$h^1 : H^1(F, \mathbb{C}) \to H^1(F, \mathbb{C})$$

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Open question : Is h^1 determined by the arrangement's combinatorics ? (Remark that h^2 can be deduced from h^1).

We have the decomposition in eigenspaces :

$$H^1(F,\mathbb{C}) = \bigoplus_{\beta \,|\, \beta^n=1} H^1(F,\mathbb{C})_\beta,$$

where $H^1(F,\mathbb{C})_\beta = \ker(h^1 - \beta \cdot Id)$.

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Fact :

$$H^1(F,\mathbb{C})_\beta = H^1(M(\mathcal{A}),\mathcal{L}_\beta),$$

where $\mathcal{L}_\beta$ is the rank one local system on the complement $M(\mathcal{A})$ such that the monodromy t_i around each hyperplane H_i is β.

The computation of local system cohomology of complement is related with :

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We can use our main result to prove the vanishing of the monodromy eigenspaces $H^1(F, \mathbb{C})_\beta = H^1(M(\mathcal{A}), \mathcal{L}_\beta)$.

Application 1

Corollary (Ceva arrangement)

The monodromy eigenspaces $H^1(F,\mathbb{C})_\beta$ of the Milnor fiber cohomology for the monomial arrangement a.k.a. the Ceva arrangement

$$\mathcal{A}(m,m,3) : (x^m - y^m)(y^m - z^m)(x^m - z^m) = 0$$

are trivial if $\beta^3 \neq 1$, for any $m \geq 3$.

Remarks :

- We can eliminate eigenvalues with A. Libgober's result.
- The eigenvalues $\beta^3 = 1$, $\beta \neq 1$ have already been determined :

  A. Măcinic, S. Papadima, C.R. Popescu, Modular equalities for complex reflexion arrangements, *Documenta Math*. **22** (2017), 135–150.

- This result was previously established in :

  A. Dimca, On the Milnor monodromy of the irreducible complex reflection arrangements, arXiv :1606.04048.

Application 2

Consider the reflection arrangement $\mathcal{A}(G_{31})$ in $\mathbb{C}^4$, defined by the equation $f = 0$, where

$$f = xyzt(x^4 - y^4)(x^4 - z^4)(x^4 - t^4)(y^4 - z^4)(y^4 - t^4)(z^4 - t^4)$$
$$((x-y)^2-(z+t)^2)((x-y)^2-(z-t)^2)((x+y)^2-(z+t)^2)((x+y)^2-(z-t)^2)$$
$$((x-y)^2+(z+t)^2)((x-y)^2+(z-t)^2)((x+y)^2+(z+t)^2)((x+y)^2+(z-t)^2)$$
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Proposition (Reflection arrangement)

Let $\mathcal{A}(G_{31})$ be the reflection arrangement in $\mathbb{C}^4$ corresponding to the exceptional group G_{31}, and let F be the associated Milnor fiber. Then the monodromy action on $H^1(F, \mathbb{C})$ is the identity.

Remark : This result was previously established in :

A. Dimca, G. Sticlaru, On the Milnor monodromy of the exceptional reflection arrangement of type G_{31}, arXiv : 1606.06615.

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Thank you very much for your attention !