TATE PROPERTIES AND MONODROMY OF HYPERPLANE ARRANGEMENTS IN FOUR VARIABLES

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Abstract. We consider the mixed Hodge structure of the rational cohomology $H^*(F_{\mathcal{A}}, \mathbb{Q})$ of the Milnor fiber $F_{\mathcal{A}}$ of a central and essential arrangement $\mathcal{A} \subset \mathbb{C}^4$. We give the equivalence between triviality of the monodromy, Tate properties, and nullity of the non integer spectrum's coefficients.

1. Introduction

Let $\mathcal{A} = \{H_1, \dots, H_d\} \subset \mathbb{C}^{n+1}$ be a central arrangement of d hyperplanes, defined by a homogeneous, degree d polynomial $Q(x_1, \dots, x_{n+1}) \in \mathbb{C}[x_1, \dots, x_{n+1}]$, with Milnor fiber $F_{\mathcal{A}} = Q^{-1}(1) \subset \mathbb{C}^{n+1}$, and intersection lattice $L(\mathcal{A})$. For any edge $X \in L(\mathcal{A})$, we note $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}$ the corresponding subarrangement. Let $L_c(\mathcal{A})$ be the set of codimension c edges in $L(\mathcal{A})$. We say that an edge X is *dense* if the corresponding subarrangement $\mathcal{A}_X$ is not reductible, that is to say that there exists no choice of coordinates on $\mathbb{C}^{n+1}$ such that $Q_X(x_1, \dots, x_{n+1}) = Q_{X1}(x_1, \dots, x_u) \cdot Q_{X2}(x_{u+1}, \dots, x_{n+1})$, for a certain $1 \leq u \leq n$, where Q_X is the defining polynomial of $\mathcal{A}_X$, Q_{X1} and Q_{X2} are two nonconstant polynomials. We will denote by $\boldsymbol{S}$ the set of dense edges excepting hyperplanes, and $\boldsymbol{S}^{(c)}$ its restriction to codimension c edges.

We associate to $\mathcal{A}$ the projective arrangement $\mathcal{A}' \subset \mathbb{P}^n_{\mathbb{C}}$ obtained by associating to a hyperplane $H \in \mathcal{A}$, given by $\ell_H = 0$, the hyperplane $H' \in \mathbb{P}^n_{\mathbb{C}}$ defined by the same equation $\ell_H = 0$. We note $M(\mathcal{A}) \subset \mathbb{C}^{n+1}$ and $M(\mathcal{A}') \subset \mathbb{P}^n_{\mathbb{C}}$ the complements of $\mathcal{A}$ and $\mathcal{A}'$.

Let $\lambda = exp(2i\pi/d)$ be a primitive $d-$root of the unity, where $i = \sqrt{-1}$. We consider the monodromy $h : F_{\mathcal{A}} \to F_{\mathcal{A}}$, given by $h(x) = \lambda \cdot x$, and the monodromy operators induced at cohomology level $h^m : H^m(F_{\mathcal{A}}, \mathbb{C}) \to H^m(F_{\mathcal{A}}, \mathbb{C})$, for $m \geq 0$. We denote by $H^m(F_{\mathcal{A}}, \mathbb{C})_\beta$ the $\beta-$eigenspace of h^m, for $\beta \in \mu_d = \{\lambda^k, 0 \leq k \leq d-1\}$.

Consider the mixed Hodge structure of the rational cohomology $H^*(F_{\mathcal{A}}, \mathbb{Q})$ of the Milnor fiber, and the mixed Hodge numbers $h^{p,q}(H^m(F_{\mathcal{A}}, \mathbb{C})) = \dim H^{p,q}(H^m(F_{\mathcal{A}}, \mathbb{C}))$. Because the monodromy is an algebraic morphism, we can consider the equivariant Hodge numbers $h^{p,q}(H^m(F_{\mathcal{A}}, \mathbb{C}))_\beta = \dim H^{p,q}(H^m(F_{\mathcal{A}}, \mathbb{C}))_\beta$, for $\beta \in \mu_d$.

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We say that a complex variety Y is *cohomologically Tate* if for any cohomology group $H^m(Y, \mathbb{C})$, one has the following vanishing of mixed Hodge numbers:

$$h^{p,q}(H^m(Y, \mathbb{C})) = 0, \text{ for } p \neq q.$$

We know that hyperplane arrangements complements are cohomologically Tate, see for instance [6], and one can ask the question to know if it is the same for the Milnor fiber of a central arrangement $\mathcal{A} \subset \mathbb{C}^{n+1}$. If h^* is trivial on all the cohomology groups $H^*(F_{\mathcal{A}}, \mathbb{C})$, then we have that $H^m(F_{\mathcal{A}}, \mathbb{C}) = H^m(M(\mathcal{A}'), \mathbb{C})$ is a pure Hodge structure of type (m, m) and $F_{\mathcal{A}}$ is cohomologically Tate. The reciprocal is true in dimension 2, by using the the mixed Hodge structure $H^*(F_{\mathcal{A}}, \mathbb{Q})$ and [5, Theorem 1.3]. In his paper [4], A. Dimca proved the reciprocal is true for central and essential arrangements in $\mathbb{C}^3$, and constructed an example of central arrangement $\mathcal{A} \subset \mathbb{C}^8$ such that the Milnor fiber is cohomologically Tate and the monodromy is not trivial, see [4, Theorem 1.1, Example 4.3]. The question is still open for central arrangements $\mathcal{A} \subset \mathbb{C}^{n+1}$, $3 \leq n \leq 6$. In this paper we propose to study the case $n = 3$.

We have seen that our question was related to the triviality of the monodromy operators h^*. We will see that it is related to an other object, called the spectrum of the arrangement. The spectrum of a central arrangement $\mathcal{A} \subset \mathbb{C}^{n+1}$ with Milnor fiber $F_{\mathcal{A}}$ is given by

$$Sp(\mathcal{A}) = \sum_{\alpha \in \mathbb{Q}} n_\alpha \cdot t^\alpha,$$

where

$$n_\alpha = \sum_m (-1)^{m-n} \dim Gr_F^p \tilde{H}^m(F_{\mathcal{A}}, \mathbb{C})_\beta, \tag{1.1}$$

with $p = \lfloor n + 1 - \alpha \rfloor$, $\beta = exp(-2i\pi\alpha)$, and $Gr_F^p \tilde{H}^m(F_{\mathcal{A}}, \mathbb{C})_\beta$ is the $\beta-$eigenspace of the linear map induced by the monodromy operator h^m on the graduate space $Gr_F^p \tilde{H}^m(F_{\mathcal{A}}, \mathbb{C})$ of degree p of the Hodge filtration F. The spectrum of a central arrangement is combinatorially determined, and we have formulas to compute the coefficients n_α for $n = 2$, due to N. Budur and M. Saito, see [1, Theorem 1, Theorem 3]. More recently, Y. Yoon gave combinatorial formulas in the case $n = 3$, [8, Theorem 1.1], that we will use to prove the following main result of this paper, Theorem 1.1.

Theorem 1.1. *Let $\mathcal{A} \subset \mathbb{C}^4$ be an essential and central arrangement of d hyperplanes, with Milnor fiber $F_{\mathcal{A}}$. The following conditions are equivalent.*

(i) The monodromy action h^* is trivial on all the cohomology groups $H^*(F_{\mathcal{A}}, \mathbb{C})$.
(ii) $F_{\mathcal{A}}$ is cohomologically Tate.
(iii) The spectral numbers n_α vanish for all $\alpha \notin \mathbb{Z}$.

2. Proof of Theorem 1.1

Let us begin by noting that there are two possible defining polynomials $Q(x, y, z, t)$ for a central and reductible arrangement $\mathcal{A} \subset \mathbb{C}^4$:

(1) Type 1: $Q(x, y, z, t) = Q_1(x, y, z) \cdot t$

(2) Type 2: $Q(x, y, z, t) = Q_1(x, y) \cdot Q_2(z, t)$,

where Q_i are nonconstant, homogeneous polynomials. To prove our main theorem, we will need the two following lemmas.

Lemma 2.1. *Let $\mathcal{A} \subset \mathbb{C}^4$ be a central arrangement of type 2. Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be its decomposition as the product of two irreductible arrangements, where $\mathcal{A}_1 : Q_1(x, y) = 0$, and $\mathcal{A}_2 : Q_2(z, t) = 0$. Let us note $d_1 = |\mathcal{A}_1|$, $d_2 = |\mathcal{A}_2|$, $d = d_1 + d_2$. Assume $d_1 \geq 3$ and $d_2 \geq 3$. Then the coefficients n_α of the spectrum $Sp(\mathcal{A})$ are given by the following formulas:*

$$n_1 = (d_1 - 1)(d_2 - 1),\ n_2 = 1 - d_1 d_2,\ n_3 = d_1 + d_2 - 1,$$

and for $1 \leq j \leq d - 1$ we have:

$$n_{j/d} = \begin{cases} 0 & \text{if } d \nmid jd_1 \\ -(\lceil jd_1/d \rceil - 1)(\lceil jd_1/d \rceil - j + 1) & \text{otherwise} \end{cases},$$

$$n_{4-j/d} = \begin{cases} 0 & \text{if } d \nmid jd_1 \\ (\lfloor jd_1/d \rfloor - 1)(\lfloor jd_1/d \rfloor - j + 1) & \text{otherwise} \end{cases},$$

$$n_{1+j/d} = \begin{cases} 0 & \text{if } d \nmid jd_1 \\ 3\lceil jd_1/d \rceil^2 + (-d_1 + d_2 - 3j)\lceil jd_1/d \rceil + (-1 + j)d_1 - d_2 + j + 1 & \text{otherwise} \end{cases},$$

$$n_{3-j/d} = \begin{cases} 0 & \text{if } d \nmid jd_1 \\ -3\lfloor jd_1/d \rfloor^2 + (d_1 - d_2 + 3j)\lfloor jd_1/d \rfloor + (1 - j)d_1 + d_2 - j - 1 & \text{otherwise} \end{cases}.$$

Proof. For such an arrangement, we easily see that $\boldsymbol{S}^{(3)} = \emptyset$ and $\boldsymbol{S}^{(2)} = \{V_1, V_2\}$, where $V_1 := \{x = y = 0\}$ and $V_2 := \{z = t = 0\}$ are two codimension 2 dense edges of multiplicity d_1 and d_2. Then we use the formulas of [8, Thorme 1.1]:

- $n_{j/d} = \binom{j-1}{3} - (j-3)\binom{\lceil jd_1/d \rceil - 1}{2} + 2\binom{\lceil jd_1/d \rceil - 1}{3} - (j-3)\binom{\lceil jd_2/d \rceil - 1}{2} + 2\binom{\lceil jd_2/d \rceil - 1}{3}$,
- $n_{1+j/d} = (d - j - 1)\binom{j-1}{2} - (\lceil jd_1/d \rceil - 1)\lfloor (d-j)d_1/d \rfloor (j-2)$
  $-(d - j - 1 - 2\lfloor (d-j)d_1/d \rfloor)\binom{\lceil jd_1/d \rceil - 1}{2} - (\lceil jd_2/d \rceil - 1)\lfloor (d-j)d_2/d \rfloor (j-2)$
  $- (d - j - 1 - 2\lfloor (d-j)d_2/d \rfloor)\binom{\lceil jd_2/d \rceil - 1}{2}$,

for $j \in \{1, \ldots, d\}$, and

- $n_{4-j/d} = \binom{j-1}{3} - (j-3)\binom{\lfloor jd_1/d \rfloor}{2} + 2\binom{\lfloor jd_1/d \rfloor}{3} - (j-3)\binom{\lfloor jd_2/d \rfloor}{2} + 2\binom{\lfloor jd_2/d \rfloor}{3} + \delta_{0,j}$,
- $n_{3-j/d} = (d - j - 1)\binom{j-1}{2} - \lfloor jd_1/d \rfloor (\lceil (d-j)d_1/d \rceil - 1)(j-2)$
  $-(d - j - 1 - 2(\lceil (d-j)d_1/d \rceil - 1))\binom{\lfloor jd_1/d \rfloor}{2}$

$$-\lfloor jd_2/d\rfloor(\lceil(d-j)d_2/d\rceil-1)(j-2)-(d-j-1-2(\lceil(d-j)d_2/d\rceil-1))\binom{\lfloor jd_2/d\rfloor}{2}.$$

for $j \in \{0, \dots, d-1\}$.

The coefficients n_1, n_2, n_3 and n_4 are obtained by taking $j = d$, and the coefficients n_3, n_4 by taking $j = 0$. Let us take $j \in \{1, \dots, d-1\}$.
We find $n_{j/d}$ by using that $\lceil jd_2/d\rceil = j - \lfloor jd_1/d\rfloor$ and factorizing by $j+1-\lceil jd_1/d\rceil - \lceil jd_2/d\rceil$.

We find $n_{4-j/d}$ by using that $\lfloor jd_2/d\rfloor = j - \lceil jd_1/d\rceil$ and factorizing by $j-1-\lfloor jd_1/d\rfloor - \lfloor jd_2/d\rfloor$.

With $\lceil(d-j)d_1/d\rceil = d_1 - \lfloor jd_1/d\rfloor$, $\lceil(d-j)d_2/d\rceil = d_2 - \lfloor jd_2/d\rfloor$, and by factorizing by $\lfloor jd_1/d\rfloor + \lfloor jd_2/d\rfloor - j + 1$, we can compute $n_{3-i/d}$.

Finally, $n_{1+j/d}$ is computed by using that $\lfloor(d-j)d_1/d\rfloor = d_1 - \lceil jd_1/d\rceil$,
$\lfloor(d-j)d_2/d\rfloor = d_2 - \lceil jd_2/d\rceil$, and factorizing by $\lceil jd_1/d\rceil + \lceil jd_2/d\rceil - j - 1$. □

The second lemma gives the existence of a nonzero coefficient n_α, $\alpha \in]0,1[$, for a certain class of type 2 central arrangements.

Lemma 2.2. *Let $\mathcal{A} \subset \mathbb{C}^4$ be a central arrangement of type 2.*
Let us note $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ its decomposition as the product of two irrductible arrangements, $d_1 = |\mathcal{A}_1|$, $d_2 = |\mathcal{A}_2|$, $d = d_1 + d_2$. Assume $d_1 \geq 3$, $d_2 \geq 3$, and $gcd(d_1, d_2) = e > 1$.
Then there exists $\alpha \in]0,1[$, such that n_α is nonzero.

Proof. We know that h^* has order e with [4, Theorem 1.4]. So, let $\beta = exp(-2i\pi k/e)$, $1 \leq k \leq e$, be a $e-$root of the unity. Let us denote by $d_1' \geq 1, d_2' \geq 1$, and $d' \geq 2$ the integers such that $d_1 = d_1' \cdot e$, $d_2 = d_2' \cdot e$, and $d = d' \cdot e$. Then $\beta = exp(-2i\pi kd'/d)$, $d' \leq kd' \leq d$, and with Lemma 2.1, we can compute the coefficient n_α, $\alpha = kd'/d$, of $Sp(\mathcal{A})$.

- If $e \geq 3$, by choosing $1 < k < e$ and applying Lemma 2.1 with $j = kd'$, $d' < j < d$, we obtain:
$$n_{kd'/d} = (kd_1' - 1)(kd_2' - 1),$$
because $kd'd_1 = kd_1'd$ is divisible by d. Furthermore $kd_1', kd_2' > 1$, so $n_{kd'/d}$ is strictly positive.
- If $e = 2$, then for $k = 1$ and $j = d'$, we find:
$$n_{d'/d} = (d_1' - 1)(d_2' - 1).$$
Furthermore, $d_1', d_2' \geq 2$. Hence $n_{d'/d}$ is strictly positive.

□

Now, let us prove our Theorem 1.1.

- $\underline{(i) \Rightarrow (ii)}$ has been discussed in the introduction.

- $\underline{(ii) \Rightarrow (i)}$: Assume $F_{\mathcal{A}}$ is cohomologically Tate. Then for all $\beta \in \mu_d$, $\beta \neq 1$, we have:
  - $H^1(F_{\mathcal{A}}, \mathbb{C})_\beta = H^{1,1}(H^1(F_{\mathcal{A}}, \mathbb{C}))_\beta = 0$
  - $H^2(F_{\mathcal{A}}, \mathbb{C})_\beta = H^{1,1}(H^2(F_{\mathcal{A}}, \mathbb{C}))_\beta$, and with Kohno's Theorem we have that $H^2(F_{\mathcal{A}}, \mathbb{C})_\beta = 0$ if β is a primitive $d-$root of the unity, see [2, Corollary 2.2].
  - $H^3(F_{\mathcal{A}}, \mathbb{C})_\beta = H^{2,2}(H^3(F_{\mathcal{A}}, \mathbb{C}))_\beta$, and $h^{2,2}(H^3(F_{\mathcal{A}}, \mathbb{C}))_\beta = -E(M(\mathcal{A}'))$ if β is a primitive $d-$root of the unity, where $E(M(\mathcal{A}'))$ is the Euler characteristic of $M(\mathcal{A}')$.

  Recall that $E(M(\mathcal{A}')) = \sum_m (-1)^m \dim H^m(F_{\mathcal{A}}, \mathbb{C})_\beta$, for any $\beta \in \mu_d$, see [3, Proposition 2.5.4 (ii), Proposition 6.4.6]. Hence, if $\beta \neq 1$ and $\alpha \in]0,1[$ satisfy $\beta = exp(-2i\pi\alpha)$, then an easy computation with formula (1.1) shows that for a central arrangement $\mathcal{A} \subset \mathbb{C}^{n+1}$ we have:

$$(-1)^n E(M(\mathcal{A}')) = \sum_{s=0}^{n} n_{\alpha+s}.$$

In our case, $n = 3$ and we have:

$$E(M(\mathcal{A}')) = -n_\alpha - n_{1+\alpha} - n_{3-\alpha} - n_{4-\alpha}. \tag{2.1}$$

Furthermore, when $F_{\mathcal{A}}$ is cohomologically Tate we have:

$$h^{1,1}(H^2(F_{\mathcal{A}}, \mathbb{C}))_\beta = -n_{3-\alpha},\ h^{2,2}(H^3(F_{\mathcal{A}}, \mathbb{C}))_\beta = n_{1+\alpha}.$$

Let us choose $\beta = \lambda = exp(2i\pi/d)$, $\alpha = (d-1)/d$. Then $\dim H^2(F_{\mathcal{A}}, \mathbb{C})_\lambda = -n_{3-\frac{d-1}{d}} = 0$ because λ is primitive. Let us consider now the formulas of [8, Theorem 1.1]:

(1) $n_{3-\frac{d-1}{d}}$
$= -\sum_{W \in \boldsymbol{S}^{(3)}} (\lceil m_W/d \rceil - 1)\binom{\lfloor (d-1)m_W/d \rfloor}{2}$
$-\sum_{V \in \boldsymbol{S}^{(2)}} \left(\lfloor (d-1)m_V/d \rfloor (\lceil m_V/d \rceil - 1)(d-3) + (2 - 2\lceil m_V/d \rceil)\binom{\lfloor (d-1)m_V/d \rfloor}{2} \right)$
$+\sum_{V \in \boldsymbol{S}^{(2)}} \sum_{\substack{W \in \boldsymbol{S}^{(3)} \\ W \subset V}} \left(\lfloor (d-1)m_V/d \rfloor (\lceil m_V/d \rceil - 1)(\lfloor (d-1)m_W/d \rfloor - \lfloor (d-1)m_V/d \rfloor) + (\lceil m_W/d \rceil - 1)\binom{\lfloor (d-1)m_V/d \rfloor}{2} \right).$

(2) $n_{1+\frac{d-1}{d}}$
$= -\sum_{W \in \boldsymbol{S}^{(3)}} \lfloor m_W/d \rfloor \binom{\lceil (d-1)m_W/d \rceil - 1}{2}$
$-\sum_{V \in \boldsymbol{S}^{(2)}} \left((\lceil (d-1)m_V/d \rceil - 1)\lfloor m_V/d \rfloor (d-3) - 2\lfloor m_V/d \rfloor \binom{\lceil (d-1)m_V/d \rceil - 1}{2} \right)$
$+\sum_{V \in \boldsymbol{S}^{(2)}} \sum_{\substack{W \in \boldsymbol{S}^{(3)} \\ W \subset V}} \left((\lceil (d-1)m_V/d \rceil - 1)\lfloor m_V/d \rfloor (\lceil (d-1)m_W/d \rceil - \lceil (d-1)m_V/d \rceil) + \lfloor m_W/d \rfloor \binom{\lceil (d-1)m_V/d \rceil - 1}{2} \right).$

By using the fact that

$$\lceil m_V/d \rceil = \lfloor m_V/d \rfloor + 1,$$

and

$$\lceil (d-1)m_V/d \rceil = \lfloor (d-1)m_V/d \rfloor + 1,$$

we easily check that $n_{3-\frac{d-1}{d}} = n_{1+\frac{d-1}{d}} = 0$.

Hence $E(M(\mathcal{A}')) = -h^{2,2}(H^3(F_{\mathcal{A}}, \mathbb{C}))_\lambda = -n_{1+\frac{d-1}{d}} = 0$ and $\mathcal{A}$ is reductible with [7, Theorem 5], that is to say that $\mathcal{A}$ is type 1 or type 2. We will study these two types separatly.

(1) If $\mathcal{A}$ is type 1, then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1 : Q_1(x,y,z) = 0$ is composed by $d-1$ hyperplanes, and $\mathcal{A}_2$ by the hyperplane $\{t = 0\}$. Hence $gcd(|\mathcal{A}_1|, |\mathcal{A}_2|) = 1$ and h^* is the identity with [4, Theorem 1.2].

(2) If $\mathcal{A}$ is type 2, then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, with $\mathcal{A}_1 : Q_1(x,y) = 0$, and $\mathcal{A}_2 : Q_2(z,t) = 0$. Let us note $d_1 = |\mathcal{A}_1|$, $d_2 = |\mathcal{A}_2|$, and $e = gcd(d_1, d_2)$.
   (a) If $e = 1$, we conclude as previously.
   (b) If $e > 1$, then with [4, Theorem 1.4] we have that h^* has order e, and for any $\beta = exp(-2i\pi k/e) \in \mu_e$, $1 \leq k \leq e$, we have $H^2(F_{\mathcal{A}}, \mathbb{C})_\beta = H^0(\mathbb{T}, \mathbb{C}) \otimes H^1(F_{\mathcal{A}_1}, \mathbb{C})_\beta \otimes H^1(F_{\mathcal{A}_2}, \mathbb{C})_\beta$, where $F_{\mathcal{A}_1}$ et $F_{\mathcal{A}_2}$ are the Milnor fibers of $\mathcal{A}_1$ and $\mathcal{A}_2$. If we note $d_1' = d_1/e$, $d_2' = d_2/e \in \mathbb{N}^*$, then $\beta = exp(-2i\pi k d_1'/d_1) = exp(-2i\pi k d_2'/d_2)$, with $d_1' \leq k d_1' \leq d_1$ and $d_2' \leq k d_2' \leq d_2$.
   If $e > 2$, then by choosing $1 < k < e$, we have $1 < k d_1' < d_1$ and $1 < k d_2' < d_2$. With [8, Corollary 1.3] we find the following dimensions: $h^{1,0}(H^1(F_{\mathcal{A}_1}, \mathbb{C}))_\beta = k d_1' - 1 > 0$, and $h^{1,0}(H^1(F_{\mathcal{A}_2}, \mathbb{C}))_\beta = k d_2' - 1 > 0$. Hence there exist $\omega_1 \in H^1(F_{\mathcal{A}_1}, \mathbb{C})_\beta$ and $\omega_2 \in H^1(F_{\mathcal{A}_2}, \mathbb{C})_\beta$. So, $H^{2,0}(H^2(F_{\mathcal{A}}, \mathbb{C}))$ contains $\omega_1 \otimes \omega_2$, which contradicts our assumptions.
   If $e = 2$, then $d_1', d_2' > 1$ and we obtain a contradiction by taking $k = 1$.

- $(i) \Rightarrow (iii)$ is obvious.
- $(iii) \Rightarrow (i)$: Assume (iii) holds. Let us take $\alpha \in]0,1[$. It follows from (2.1) and [7, Theorem 5] that $\mathcal{A}$ is reductible. If $\mathcal{A}$ is type 1, or type 2 with $gcd(d_1, d_2) = 1$, then the action of the monodromy h^* is trivial.
The cases where $\mathcal{A}$ is type 2 and $d_1 = 2$ or $d_2 = 2$ correspond to a type 1 arrangement.
Finally, the case where $\mathcal{A}$ is type 2 and $pgcd(d_1, d_2) > 1$, $d_1, d_2 \geq 3$, is excluded with Lemma 2.2.

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