Tate properties and monodromy of hyperplane arrangements

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Let $\mathcal{A} = \{H_1, \dots, H_d\} \subset \mathbb{C}^{n+1}$ be a **hyperplane arrangement**.

We say that $\mathcal{A}$ is **central** if $0 \in \cap_{j=1}^{d} H_j$.

We say that $\mathcal{A}$ est **essential** if $\cap_{j=1}^{d} H_j = \{0\}$.

The union of hyperplanes of $\mathcal{A}$ is defined by a polynomial :

$$Q(x_1, \dots, x_{n+1}) = \prod_{j=1}^{d} l_j(x_1, \dots, x_{n+1}) = 0,$$

where $H_j := \{l_j = 0\}$.

When $\mathcal{A}$ is central, Q is homogeneous (of degree d) and we consider the **projective arrangement** :

$$\mathcal{A}' = \{H'_1, \dots, H'_d\} \subset \mathbb{P}^n_{\mathbb{C}}.$$

Let us note

$$M(\mathcal{A}) = \mathbb{C}^{n+1} \setminus \bigcup_{j=1}^{d} H_j, \quad M(\mathcal{A}') = \mathbb{P}^n_{\mathbb{C}} \setminus \bigcup_{j=1}^{d} H'_j,$$

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In this talk, we'll only consider central arrangements.

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Theorem (Orlik, Solomon, 1980)

Let R be a commutative unitary ring. Then $H^*(M(\mathcal{A}), R)$ is completely determined by $L(\mathcal{A})$.

Let $\mathcal{A} = \{H_1, \dots, H_d\} \subset \mathbb{C}^{n+1}$ **central**.

Definition

The **Milnor fiber** of $\mathcal{A}$ is the smooth, affine hypersurface defined by

$$F = Q^{-1}(1) \subset \mathbb{C}^{n+1}.$$

Let $\lambda = exp(2i\pi/d)$, $d = |\mathcal{A}|$.

The **monodromy** on F is the map

$$\begin{aligned} h: \ & F \to F \\ & x \mapsto \lambda \cdot x \end{aligned}$$

We note

$$h^q : H^q(F,\mathbb{C}) \to H^q(F,\mathbb{C}), \quad \forall q \geq 0,$$

the **monodromy operators**, with eigenvalues in

$$\mu_d = \{\lambda^k, \, 0 \leq k \leq d-1\},$$

and

$$H^q(F,\mathbb{C})_\beta$$

the $\beta-$eigenspace, for any $\beta \in \mu_d$.

OPEN QUESTION :

$$L(\mathcal{A}) \Longrightarrow h^* \ ?$$

$$L(\mathcal{A}) \Longrightarrow b_q(F) = \dim H^q(F, \mathbb{C}), \quad 0 \leq q \leq n \ ?$$

Related to many things. We can use mixed Hodge theory.

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(Deligne) $\Rightarrow H^q(F, \mathbb{Q})$ is a mixed Hodge structure for all $q \geq 0$.

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- There exists a finite, increasing filtration, called "weight filtration"

$$0 \subset W_qH^q(F,\mathbb{Q}) \subset \cdots \subset W_{2q}H^q(F,\mathbb{Q}) = H^q(F,\mathbb{Q}),$$

such that

$$Gr_k^W H^q(F,\mathbb{Q}) = W_kH^q(F,\mathbb{Q})/W_{k-1}H^q(F,\mathbb{Q})$$

is a pure Hodge structure of weight k for all k.

There exists an induced filtration,

$$F^a Gr_k^W H^q(F,\mathbb{C})$$

and we note

$$H^{a,b}(H^q(F,\mathbb{C})) = Gr_F^a\, Gr_{a+b}^W\, H^q(F,\mathbb{C}).$$

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Definition

The **mixed Hodge numbers** are the dimensions

$$h^{a,b}(H^q(F, \mathbb{C})) = \dim_{\mathbb{C}} H^{a,b}(H^q(F, \mathbb{C})).$$

One can try to compute these mixed Hodge numbers using the monodromy h. We are interesting today in a property of these numbers.

Definition

Let Y be a complex algebraic variety. We say that Y is **cohomologically Tate** if one has the following vanishing of mixed Hodge numbers :

$$h^{a,b}(H^q(Y, \mathbb{C})) = 0, \quad \text{if } a \neq b, \ \forall q.$$

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QUESTION : Is the Milnor fiber F of a central hyperplane arrangement $\mathcal{A}$ cohomologically Tate ?

- $M(\mathcal{A})$ is cohomologically Tate :

  $H^q(M(\mathcal{A}), \mathbb{C})$ is a pure Hodge structure of type (q, q).

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- h^* trivial $\Rightarrow \; H^*(F, \mathbb{C}) = H^*(F, \mathbb{C})_1 = H^*(M(\mathcal{A}'), \mathbb{C})$

  $\Rightarrow$ F cohomologically Tate.

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- TRUE for $\mathcal{A} \subset \mathbb{C}^2$ central.
- (Dimca, 2012) TRUE for $\mathcal{A} \subset \mathbb{C}^3$ central and essential.
- (Dimca, 2012) False pour $\mathcal{A} \subset \mathbb{C}^8$.

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Today we study the case $\mathcal{A} \subset \mathbb{C}^4$ central.

We consider

$$Gr_F^a H^q(F, \mathbb{C}),$$

and the induced linear map :

$$h^q : Gr_F^a H^q(F, \mathbb{C}) \to Gr_F^a H^q(F, \mathbb{C}),$$

with eigenvalues $\in \mu_d$.

Let $\beta \in \mu_d$. We note

$$Gr_F^a H^q(F, \mathbb{C})_\beta$$

the $\beta-$eigenspace.

Definition

The **spectrum** of a central arrangement $\mathcal{A} \subset \mathbb{C}^{n+1}$, with Milnor fiber F, is the polynomial

$$Sp(\mathcal{A}) = \sum_{\alpha \in \mathbb{Q}} n_\alpha \; t^\alpha,$$

where the coefficients are given by :

$$n_\alpha = \sum_{q>0} (-1)^{q-n} \dim Gr_F^a H^q(F, \mathbb{C})_\beta,$$

where $\beta = exp(-2i\pi\alpha)$, and $a = \lfloor n+1-\alpha \rfloor$.

Theorem (Budur, Saito, 2009)

The spectrum $Sp(\mathcal{A})$ of a central arrangement $\mathcal{A} \subset \mathbb{C}^{n+1}$ is completely determined by $L(\mathcal{A})$.

Hence we have hope to express the monodromy and the cohomology groups $H^q(F, \mathbb{C})$ in terms of the arrangement's combinatorics.

- Budur, Saito : we have formulas to compute the spectrum coefficients n_α for $\mathcal{A} \subset \mathbb{C}^3$ central and essential.
- Yoon, 2014 : new formulas for $\mathcal{A} \subset \mathbb{C}^4$ central.

Theorem

Let $\mathcal{A} \subset \mathbb{C}^4$ be a central and essential arrangement, with Milnor fiber F. Are equivalent :

(i) h^* is trivial on all the cohomology groups $H^*(F, \mathbb{C})$.

(ii) F is cohomologically Tate.

(iii) The spectrum coefficients n_α vanish for all $\alpha \notin \mathbb{Z}$.

For $i \in \{1, \dots, d\}$:

$$n_{\frac{i}{d}} = \binom{i-1}{3} - \sum_{W \in \boldsymbol{S}^{(3)}} \binom{\lceil i \cdot m_W/d \rceil - 1}{3}$$

$$- \sum_{V \in \boldsymbol{S}^{(2)}} \left((i-3)\binom{\lceil i \cdot m_V/d \rceil - 1}{2} - 2\binom{\lceil i \cdot m_V/d \rceil - 1}{3} \right)$$

$$- \sum_{V \in \boldsymbol{S}^{(2)}} \sum_{\substack{W \in \boldsymbol{S}^{(3)} \\ W \subset V}} \left(2\binom{\lceil i \cdot m_V/d \rceil - 1}{3} - (\lfloor i \cdot m_W/d \rfloor - 3)\binom{\lceil i \cdot m_V/d \rceil - 1}{2} \right)$$

$$+\delta_{0,i}$$

Thank you for your attention !